Transverse quasilinear relaxation in an inhomogeneous magnetic field

Maxim Lyutikov
Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125
and CITA, Toronto, Canada M5S 3M8
(Received 26 January 1998)

Transverse quasilinear relaxation of the cyclotron Cherenkov instability of an ultrarelativistic beam propagating along a strong, inhomogeneous magnetic field in a pair plasma is considered. We find a quasilinear state in which the kinetic-type instability is saturated by the force arising in the inhomogeneous field due to the conservation of the adiabatic invariant. The resulting wave intensities generally have a non-power-law frequency dependence, but in a broad frequency range can be well approximated by a power law with a spectral index -2. The emergent spectra and fluxes are consistent with the one observed from radio pulsars. [S1063-651X(98)14108-9]

PACS number(s): 52.40.Mj, 52.25.Dg, 52.25.Sw, 97.60.Gb

I. INTRODUCTION

Unusual physical conditions present in pulsar magnetospheres (a superstrong inhomogeneous magnetic field populated by a relativistic one-dimensional pair plasma, and penetrated by the ultrarelativistic $\gamma \approx 10^7$ beam; see, e.g., Ref. [1]), require a consideration of the known physical processes in the new parameter domain. This may serve as an important step in identifying a pulsar radio emission mechanism: a problem that remains unsolved in spite of 30 years of intensive research. In this paper we consider quasilinear relaxation in an inhomogeneous magnetic field of a highly relativistic beam propagating along a strong magnetic field through a pair plasma. A beam may undergo a cyclotron Cherenkov instability which may be responsible for the generation of the observed emission [2–5]. The cyclotron Cherenkov instability develops at the anomalous Doppler resonance

$$\omega(\mathbf{k}) - k_{\|} v_{\|} - s\frac{\omega_B}{\gamma} = 0, \tag{1}$$

with $s<0$. In Eq. (1), $\omega(\mathbf{k})$ is the frequency of the normal mode, $\mathbf{k}$ is a wave vector, v is the velocity of the resonant particle, $\omega_B = |q|B/mc$ is the nonrelativistic gyrofrequency, γ is the Lorentz factor in the pulsar frame, q is the charge of the resonant particle, m is its mass, and c is the speed of light. It has been shown (e.g., Ref. [5]), that the cyclotron Cherenkov instability can explain a broad variety of the observed pulsar phenomena.

Close to the stellar surface, where the initial beam is produced and accelerated, the particles quickly reach their ground gyrational state due to synchrotron emission in a superstrong magnetic field, so that their distribution becomes virtually one dimensional [1]. In the outer parts of the magnetosphere it becomes possible to satisfy the anomalous Doppler resonance—the cyclotron Cherenkov instability develops, bringing about the diffusion of particles in transverse moments. The relevant saturation mechanism then determines the final spectrum (which can be later modified to be the absorption processes).

The nonlinear saturation of the cyclotron Cherenkov instability due to the diffusion of the resonant particles was previously considered by several authors. Kawamura and Suzuki [2] neglected the possible stabilizing effects of the radiation reaction force due to the cyclotron emission at the normal Doppler resonance, and the force arising in the inhomogeneous magnetic field due to the conservation of the adiabatic invariant. These forces result in a saturation of the quasilinear diffusion.

The authors of Ref. [3] were the first to notice the importance of the radiation reaction force due to the emission at the normal Doppler resonance on the saturation of the quasilinear diffusion. Unfortunately, they [3] used an expression for the cyclotron damping rates which is applicable only for the nonrelativistic transverse motions, when $\gamma\psi$ (ψ is the pitch angle) is much less than unity. In pulsar magnetospheres the development of the cyclotron Cherenkov instability results in a diffusion of particles in transverse moments, quickly increasing the transverse energy there to relativistic values.

In a review paper, the authors of Ref. [6] took a correct account of the radiation reaction force due to the emission at the normal Doppler resonance, and pointed out the importance of the force arising in the inhomogeneous magnetic field due to the conservation of the adiabatic invariant [**G** force, Eq. (5)]. When considering the deceleration of the beam the authors of Ref. [6] incorrectly neglected the radiation reaction force due to the emission at the anomalous Doppler resonance in comparison with the radiation reaction force due to the emission at the normal Doppler resonance.

In this work we reconsider and extend the treatment of the quasilinear stage of the cyclotron Cherenkov instability in the inhomogeneous magnetic field. We found a state in which the particles are constantly slowing down their parallel motion, mainly due to the component along the magnetic field of the radiation reaction force of emission at the anomalous Doppler resonance. At the same time the particles retain a pitch angle almost constant due to the balance of the force $G_\perp$ and the component perpendicular to the magnetic field of the radiation reaction force of emission at the *anomalous* Doppler resonance. We calculate the distribution function and the wave intensities for such a quasilinear state.

TABLE I. Dimensions of the main quantities used.

$E^2(\mathbf{k})$	E_k^2	$n(\mathbf{k})$	$n(k)$	$\alpha,\ \beta$	$D_{\psi\psi}$	$D_{\psi p}$	D_{pp}	D	A	$f(p)p^2dp,\ f(\gamma)d\gamma$
erg	erg/cm^2	1	1/cm^2	erg/cm	1/s	erg/cm	erg^2 s/cm^2	cm^2/erg s	cm	1

In the process of quasilinear diffusion, the initial beam loses a large fraction of its initial energy $\approx$10%, which is enough to explain the typical luminosities of pulsars. Though the quasisteady wave intensities are not strictly power laws [see Eq. (46)], they can be well approximated by a power law with a spectral index $F(\nu)\propto^{-2}$ [$F(\nu)$ is the spectral flux density in Janskys] which is very close to the observed mean spectral index of -1.6 [7]. The predicted spectra also show a turn off at the low frequencies $\nu\leqslant300$ MHz and a flattering of spectrum at large frequencies $\nu\geqslant1$ GHz which may be related to the possible turn up in the flux densities at mm wavelengths [8].

In Sec. II, we derive a kinetic equation which describes the wave-particle interaction in the strong magnetized, weakly inhomogeneous plasma, under the assumption that the particle-photon collisions are described by the Fokker-Plank terms (particle-particle collisions are neglected). This kinetic equation is used in Sec. III to find a quasilinear state. We find the particle distribution function (which is not a power law) and the emergent spectrum. As a reference, in Table I we give the dimensions of the main quantities used.

II. DRIFT KINETIC EQUATION WITH RADIATION DAMPING

We adopt the view that the plasma is described by a distribution function $f(\mathbf{r},\mathbf{p},t)$ whose evolution follows Liouville's theorem (the continuity equation in six-dimensional phase space).

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial \mathbf{r}}[\mathbf{v}f]+\frac{\partial}{\partial \mathbf{p}}\left[\frac{d\mathbf{p}}{dt}f\right]=\left.\frac{\partial f}{\partial t}\right|_{\rm coll}, \tag{2}$$

where $\mathbf{v}=d\mathbf{r}/dt$. Equation (2) implies that the density of particles in a unit volume of the phase space changes only due to collisions.

In this section we modify Eq. (2) under three separate assumptions. First, we assume that the Larmor radius of the particle gyration is much smaller than the inhomogeneity scale of the static magnetic field. This allows us to use a drift approximation (e.g., Ref. [10]) in which a motion of a particle consists of the slow drift motion of the guiding center and fast gyration around the local magnetic field. As the particle propagates in an inhomogeneous magnetic field the projection of the momentum on the local magnetic field and the amplitude of fast gyrations will change to conserve the first adiabatic invariant. These slow changes may be written in terms of an effective, dissipative, i.e, momentum-dependent, force $\mathbf{G}$.

Second, we assume that the particle-particle collisions are unimportant. Third, we assume that the interaction of charged particles with the electromagnetic waves can be described by the Fokker-Plank collision terms. We will show that a stochastic force due to *spontaneous* emission, which is conventionally included in the "drift" Fokker-Plank terms on the right-hand side of Eq. (2), may be written as a radiation damping force on the left-hand side of the corresponding kinetic equation [see Eq. (18)].

A. Drift approximation

The equations of motion of a particle in a weakly inhomogeneous static magnetic field $\mathbf{B}_0$ are

$$\frac{d\mathbf{p}}{dt}=\frac{q}{c}\mathbf{v}\times\mathbf{B}_0\,. \tag{3}$$

[Note that the forces due to the interaction of a charged particle with a fast changing electromagnetic field are assumed to be incorporated in the Fokker-Plank terms on the right-hand side of Eq. (2).] The condition of weak inhomogeneity allows a simplification of Eq. (3) in the drift approximation (e.g., Ref. [10]):

$$\frac{dp_\parallel}{dt}=G_\parallel\,, \qquad \frac{dp_\perp}{dt}=G_\perp\,, \tag{4}$$

where $p_\parallel=\mathbf{p}\cdot\mathbf{B}_0/B_0$ is the momentum along the local field $\mathbf{B}_0$, $p_\perp=|\mathbf{p}-p_\parallel\mathbf{B}_0/B_0|$ is a momentum perpendicular to the local field $\mathbf{B}_0$, and $\mathbf{G}$ is a force due to the inhomogeneity of the magnetic field $\mathbf{B}_0$. Its components are

$$G_\parallel=-\beta\gamma\psi^2,\quad G_\perp=-\beta\gamma\psi,\quad \beta=\frac{mc^2}{2R_B}. \tag{5}$$

Here $R_B\approx10^9$ cm is the radius of curvature of the magnetic field. The force $\mathbf{G}$ may be thought of as arising from the conservation of the first adiabatic invariant.

Using Eq. (4), the kinetic equation (2) becomes

$$\frac{\partial f}{\partial t}+(\mathbf{v}\nabla)f+\frac{\partial}{\partial \mathbf{p}}\left[\left(\frac{q}{c}\mathbf{v}\times\mathbf{B}_0+\mathbf{G}\right)f\right]=\left.\frac{\partial f}{\partial t}\right|_{\rm coll}, \tag{6}$$

where we also used independence of $\mathbf{v}$ on $\mathbf{r}$.

B. Wave-particle interaction

We assume that the collision term on the right-hand side of Eq. (6) is described by Fokker-Plank-type terms due to the interaction of the particles with wave quanta. The Fokker-Plank equations describing the quasilinear diffusion in the magnetic field are

$$\left.\frac{\partial f}{\partial t}\right|_{\rm coll}=\frac{1}{\sin\,\psi}\frac{\partial}{\partial\psi}\left[\sin\,\psi\left(A_{\psi}+D_{\psi\psi}\frac{\partial}{\partial\psi}+D_{\psi p}\frac{\partial}{\partial p}\right)f(\mathbf{p})\right]\frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(A_p+D_{p\psi}\frac{\partial}{\partial\psi}+D_{pp}\frac{\partial}{\partial p}\right)f(\mathbf{p})\right] \tag{7}$$

$$\begin{pmatrix} D_{\psi\psi} \\ D_{\psi p}=D_{p\psi} \\ D_{pp}\end{pmatrix}=\sum_{s<0}\int\frac{d\mathbf{k}}{(2\pi)^3}w(s,\mathbf{p},\mathbf{k})n(\mathbf{k})\begin{pmatrix}(\Delta\psi)^2\\ (\Delta\psi)(\Delta p)\\ (\Delta p)^2\end{pmatrix} \tag{8}$$

$$\begin{pmatrix}A_p\\ A_{\psi}\end{pmatrix}=\sum_s\int\frac{d\mathbf{k}}{(2\pi)^3}w(s,\mathbf{p},\mathbf{k})\begin{pmatrix}\Delta p\\ \Delta\psi\end{pmatrix} \tag{9}$$

$$\frac{dn(\mathbf{k})}{dt}=\sum_{s<0}\int d\mathbf{p}\,w(s,\mathbf{p},\mathbf{k})\left[n(\mathbf{k})\hbar\left(\frac{\partial}{\partial p}+\frac{\cos\,\psi-(kv/\omega)\cos\,\theta}{p\,\sin\,\psi}\frac{\partial}{\partial\psi}\right)f(\mathbf{p})\right], \tag{10}$$

where

$$\Delta p=\frac{\hbar\omega}{v},\quad \Delta\psi=\frac{\hbar(\omega\,\cos\,\psi-k_{\parallel}v)}{pv\,\sin\,\psi}, \tag{11}$$

$$n(\mathbf{k})=\frac{E^2(\mathbf{k})}{\hbar\omega(\mathbf{k})}, \tag{12}$$

$$w(s,\mathbf{p},\mathbf{k})=\frac{8\pi^2q^2R_E(\mathbf{k})}{\hbar\omega(\mathbf{k})}|\mathbf{e}(\mathbf{k})\cdot\mathbf{V}(s,\mathbf{p},\mathbf{k})|^2 \times\delta(\omega(\mathbf{k})-s\omega_B/\gamma-k_{\parallel}v_{\parallel}), \tag{13}$$

$$\mathbf{V}(s,\mathbf{p},\mathbf{k})=\left(v_{\perp}\frac{s}{z}J_s(z),\ -i\sigma s v_{\perp}J_s(z)',v_{\parallel}J_s(z)\right). \tag{14}$$

Here $E^2(\mathbf{k})d\mathbf{k}/(2\pi)^3$ is the energy density of the waves in the unit element range of $\mathbf{k}$ space, $z=k_{\perp}p\,\sin\,\psi/|q|B_0$, J_s are Bessel functions, and σ is the sign of q. The drift Fokker-Plank terms may be associated with spontaneous emission, while the diffusive Fokker-Plank terms are associated with the induced emission.

A particle moving with a relativistic velocity in a medium interacts resonantly with the waves satisfying resonant condition given by Eq. (1). In what follows, we consider a particular case when the resonant particles move in a dielectric medium in magnetic field with the velocity larger than the velocity of light in a medium. Then such particles interact with electromagnetic waves at the anomalous Doppler resonance [$s<0$ in Eq. (1)] and at the normal Doppler resonance [$s>0$ in Eq. (1)]. We will make a distinction between four different processes described by various terms in Eq. (7): spontaneous emission at the normal Doppler effect (drift terms with $s>0$), spontaneous emission at the anomalous Doppler effect (drift terms with $s<0$), induced emission at the normal Doppler effect (diffusive terms with $s>0$) and induced emission at the at the anomalous Doppler effect (diffusive terms with $s<0$). This separation is done because the recoils that an electron experiences during emission at the normal and anomalous Doppler resonances are quite different. The radiation reaction due to the emission at the normal Doppler resonance slows the particle's motion along magnetic field and decreases its transverse momentum, while the radiation reaction due to the emission at the anomalous Doppler resonance *increases* its transverse momentum and also slows the particle's motion along magnetic field [9]. Another important difference between the emission at the normal and anomalous Doppler resonances is that the corresponding waves have considerably different frequencies. An important assumption that we will use is that at frequencies corresponding to normal Doppler resonances the influence of a medium can be neglected, so that processes with $(s>0)$ can be treated as occurring in vacuum. We also assume that initially there is no strong radiation satisfying normal Doppler resonance, so that the induced processes at the normal Doppler effect can be neglected.

Emission at the normal Doppler occur on very high frequencies for which $\omega\approx kc$. The change in the angle is then

$$\Delta\psi=\frac{s\hbar\omega_B/\gamma-kv\,\sin^2\psi}{pv\,\sin\,\psi}\approx\frac{s\hbar\omega_B}{\gamma pv\psi},\quad s>0. \tag{15}$$

Since we expect that $\gamma\psi\gg1$, it is the large s that will be important for the normal Doppler effect. Then, using the known single particle emissivity at the normal Doppler effect in vacuum, the drift Fokker-Plank terms on the right-hand side of Eq. (6) may be written as a momentum-dependent external force acting on an electron:

$$\mathbf{F}=F_{\parallel}\mathbf{e}_{\parallel}+F_{\perp}\mathbf{e}_r,\quad F_{\parallel}=F_{\parallel}^{(x)}+F_{\parallel}^{(o)},\quad F_{\perp}=F_{\perp}^{(x)}+F_{\perp}^{(o)}$$

$$F_{\parallel}^{(x)}=-\frac{e^2v_{\perp}^2v_{\parallel}}{2\pi c^4}\sum_{s=1}^{s=\infty}\int d\Omega\ d\omega\ n^2\omega^2\cos\,\theta J_s'(\lambda)^2 \times\delta(\omega-s\omega_B/\gamma-k_{\parallel}v_{\parallel}),$$

$$F_{\perp}^{(x)}=-\frac{e^2v_{\perp}^2s\omega_B/\gamma}{2\pi c^3}\sum_{s=1}^{s=\infty}\int d\Omega\ d\omega\ n\omega J_s'(\lambda)^2 \times\delta(\omega-s\omega_B/\gamma-k_{\parallel}v_{\parallel}),$$

$$F_{\parallel}^{(o)}=-\frac{e^2\, v_{\parallel}}{2\,\pi\, c^4}\sum_{s=1}^{s=\infty}\int d\Omega\; d\omega\; n\omega^2\left(nv_{\perp}\frac{s}{\lambda}-c\,\sin\,\theta\right)\times\left(\frac{s}{\lambda}\,v_{\perp}\cos\,\theta-v_{\parallel}\sin\,\theta\right)J_s^2(\lambda)\times\delta(\omega-s\omega_B/\gamma-k_{\parallel}v_{\parallel}),$$

$$F_{\perp}^{(o)}=-\frac{e^2 v_{\perp}}{2\pi c^4}\sum_{s=1}^{s=\infty}\int d\Omega\; d\omega\;\omega^2\,\frac{s}{\lambda}\,\frac{(nv_{\parallel}-c\,\cos\,\theta)^2}{\sin\,\theta}\times J_s(\lambda)^2\,\delta(\omega-s\omega_B/\gamma-k_{\parallel}v_{\parallel}), \tag{16}$$

Performing the integrations we find the classical expressions for the radiation reaction force due to the spontaneous synchrotron emission at the normal Doppler resonance:

$$F_{\parallel}=-\alpha\gamma^2\psi^2,\quad F_{\perp}=-\alpha\psi(1+\gamma^2\psi^2),\quad \alpha=\frac{2q^2\omega_B^2}{3c^2}. \tag{17}$$

This identification allows us to treat the effects of the emission at the normal Doppler effect as a dissipative force in the Boltzmann equation, and not as a drift Fokker-Plank term. Another simplification that can be made is that it is possible to ignore the spontaneous emission at the anomalous Doppler effect, since the induced emission will be enhanced by a large number of quanta satisfying anomalous Doppler resonance.

Thus of the four different stochastic processes that contributed to the Fokker-Plank terms, only the induced emission at the anomalous Doppler resonance is left on the right-hand side of Eq. (6) which becomes

$$\frac{\partial f(\mathbf{p})}{\partial t}+\mathbf{v}\frac{\partial f(\mathbf{p})}{\partial \mathbf{r}}+\frac{\partial}{\partial \mathbf{p}}\left[\left(\mathbf{G}+\mathbf{F}+\frac{q}{c}(\mathbf{v}\times\mathbf{B}_0)\right)f(\mathbf{p})\right]=\left.\frac{\partial f}{\partial t}\right|_{\mathrm{coll}^{(a)}}, \tag{18}$$

where the supersript *(a)* implies that only diffusion terms due to the resonant interaction at the anomalous Doppler resonance are retained.

III. QUASISTEADY STATE

As the particle propagates into the region of lower magnetic field, both force **G** and force **F** act to decrease its transverse momentum. The radiation reaction due to the development of the cyclotron instability at the anomalous Doppler effects tend to increase the particle's transverse momentum. A stationary state in transverse moments may be reached when the actions of the **G** force and radiation reaction due to the emission at the normal Doppler resonance is balanced by the radiation reaction due to the emission at the anomalous Doppler resonance.

From the quantum point of view, the action of the radiation reaction forces induces transitions between the different quantum states, while the force **G** changes the energy of the quantum level. Transitions up in quantum levels occur between the two adjacent levels ($s=-1$ transitions), but their rate is greatly enhanced due to the development of the instability at the anomalous Doppler effect. The transitions down in quantum levels involves a single jump in many levels ($s\approx\gamma_{\perp}^3\gg 1$ for $\gamma_{\perp}\gg 1$).

From Eqs. (5) and (19) we find that

$$\frac{F_{\parallel}}{G_{\parallel}}=\frac{\alpha}{\beta}\gamma,\quad \frac{F_{\perp}}{G_{\perp}}=\frac{\alpha}{\beta}\gamma\psi^2,\quad \text{for } \gamma\psi\gg 1, \tag{19}$$

where $r_L=c/\omega_B$ is a Larmor radius, and $r_e=q^2/(mc^2)=2.8\times 10^{-13}$ cm is a classical radius of an electron.

The dimensionless ration in Eq. (19) is

$$\frac{\alpha}{\beta}=\frac{4R_B r_e}{3r_L^2}=5\times 10^{-4}R_{B,9}R_9^{-6}. \tag{20}$$

$R_{B,9}=R_B/10^9$ cm is the radius of curvature in units of 10^9 cm, and $R_9=R/10^9$ cm is the distance from the neutron star surface in units of 10^9 cm.

Using Eq. (20), we find that for the primary particles with $\gamma\approx 10^7$

$$\frac{F_{\parallel}}{G_{\parallel}}\gg 1,\qquad \frac{F_{\perp}}{G_{\perp}}\ll 1,\quad \text{for } \psi\ll\sqrt{\frac{r_L^2}{2R_B r_e\gamma}}\approx 10^{-2}. \tag{21}$$

Neglecting $G_{\parallel}$ and $F_{\perp}$, the left-hand side of Eq. (18) may be written as

$$\frac{\partial f(\mathbf{p})}{\partial t}+\mathbf{v}\frac{\partial f(\mathbf{p})}{\partial \mathbf{r}}+\frac{1}{p\,\sin\,\psi}\frac{\partial}{\partial\psi}(\sin\,\psi G_{\perp}f(\mathbf{p}))+\frac{1}{p^2}\frac{\partial}{\partial p}(p^2F_{\parallel}f(\mathbf{p})). \tag{22}$$

In the strongly magnetized plasma of the pulsar magnetosphere, the refractive index has a strong dependence on the angle of propagation of the waves. An anomalous Doppler resonance is possible only for waves propagating inside a cone $\theta\leq\omega_p/\omega_B\ll 1$ (ω_p is the plasma frequency). This allows us to consider only waves propagating along a magnetic field. The polarization vector of such modes can be chosen as $\mathbf{e}(\mathbf{k})=(1,0,0)$. The condition of small angle propagation also allows an expansion of the Bessel functions in the transition currents (14) in small arguments $z\ll 1$, keeping only $s=-1$ terms: $\mathbf{V}(-1,\mathbf{p},\mathbf{k})\approx v_{\perp}/2(1,i\sigma,0)$. The single particle probability of emission then becomes

$$w(\pm 1,\mathbf{p},\mathbf{k})=\frac{\pi^2q^2v_{\perp}^2}{\hbar\,\omega(\mathbf{k})}\,\delta(\omega(\mathbf{k})+\omega_B/\gamma-k_{\parallel}v_{\parallel}), \tag{23}$$

where we took into account that $R_E(\mathbf{k})\approx 1/2$.

We now can find the diffusion coefficients in the approximation of a one-dimensional spectrum of the waves.

$$n(\mathbf{k})=\frac{2\pi\delta(\theta)}{k^2\sin\,\theta}n(k),\quad n(k)=\int d\mathbf{\Omega}_{\mathbf{k}}\frac{k^2}{(2\pi)^2 n(\mathbf{k})}. \tag{24}$$

The normal modes of relativistic pair plasma for the case of parallel propagation consist of two transverse waves with the dispersion relation

$$\omega = kc(1-\delta), \quad \delta = \frac{\omega_p^2 T_p}{\omega_B^2} \tag{25}$$

where T_p is the plasma temperature in units mc^2. Using Eq. (25) we can simplify the change in the pitch angle (11) in the limit $\psi^2 \ll \delta$ and $1/\gamma^2 \ll \delta$

$$\Delta\psi \approx -\frac{\hbar\omega\delta}{pv\,\sin\,\psi} \tag{26}$$

We then find

$$\begin{pmatrix} D_{\psi\psi} \\ D_{\psi p}=D_{p\psi} \\ D_{pp} \end{pmatrix} = \begin{pmatrix} D\frac{\delta}{\gamma^2}E_k^2\Big|_{k=k_{\rm res}} \\ -D\frac{\psi mc}{\gamma}E_k^2\Big|_{k=k_{\rm res}} \\ D\frac{\psi^2 m^2 c^2}{\delta}E_k^2\Big|_{k=k_{\rm res}} \end{pmatrix}, \tag{27}$$

$$D = \frac{\pi^2 q^2}{m^2 c^3} = \frac{\pi^2 r_e}{mc}$$

where

$$E_k^2 = \hbar\omega(k)n(k) = \int \frac{k^2 d\mathbf{\Omega}}{(2\pi)^2}\hbar\omega(\mathbf{k})n(\mathbf{k}) \tag{28}$$

is the energy density per unit of a one-dimensional wave vector and we assumed that $\omega(\mathbf{k})$ is an isotropic function of $\mathbf{k}$.

We next solve the partial differential equation describing the evolution of the distribution function by successive approximations. We first expand Eq. (7) in small ψ assuming that $\partial/\partial\psi \simeq 1/\psi$. We also neglect the convection term, assuming that the characteristic time for the development of the quasilinear diffusion is much smaller that the dynamic time of the plasma flow. Then we assume that it is possible to separate the distribution function into parts depending on ψ and p:

$$f(\mathbf{p}) = Y(\psi,p)f(p), \tag{29}$$

with

$$f(p) = 2\pi\int \sin\,\psi\; d\psi\; f(\mathbf{p}), \quad \int dp\, p^2 f(p) = 1. \tag{30}$$

In the lowest order in ψ we obtain an equation that describes the diffusion in pitch angles:

$$-\frac{1}{p\,\sin\,\psi}\frac{\partial}{\partial\psi}(\sin\,\psi G_\perp Y(\psi)) = \frac{1}{\sin\,\psi}\frac{\partial}{\partial\psi}\left[\sin\,\psi D_{\psi\psi}\frac{\partial Y(\psi)}{\partial\psi}\right], \tag{31}$$

which has a solution

$$Y(\psi) = \frac{1}{\pi\psi_0^2}\exp\left(-\frac{\psi^2}{\psi_0^2}\right),$$
$$\psi_0^2 = \frac{Dmc\delta E_k^2}{\beta\gamma^2} = \frac{DR_B\delta E_k^2}{c\gamma^2} = \frac{\pi^2\delta R_B r_e E_k^2}{\gamma^2 mc^2}. \tag{32}$$

The next order in ψ gives

$$\frac{\partial f(\mathbf{p})}{\partial t} + \frac{\partial}{\partial p}(F_\parallel f(\mathbf{p})) = \frac{1}{\sin\,\psi}\frac{\partial}{\partial\psi}\left[\sin\,\psi D_{\psi p}\frac{\partial f(\mathbf{p})}{\partial p}\right] + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D_{p\psi}\frac{\partial f(\mathbf{p})}{\partial\psi}\right]. \tag{33}$$

By integrating Eq. (33) over ψ with a weight ψ, we find the equation for the parallel distribution function:

$$\frac{\partial f(p)}{\partial t} - \frac{\partial}{\partial p}(AE_k^2\gamma^2 f(p)) = \frac{2}{p^2}\frac{\partial}{\partial p}(pDm^2c^2E_k^2 f(p)), \tag{34}$$

where

$$A = \frac{\alpha\psi_0^2}{E_k^2} = \frac{2q^2\omega_B^2\psi_0^2}{3c^2E_k^2} = \frac{2\pi^2\omega_B^2 q^4 R_B\delta}{3\gamma^2 m^2 c^6} = \frac{2\pi^2 R_B r_e^2\delta}{3\gamma^2 r_L^2}. \tag{35}$$

The term containing A describes the slowing of the particles due to the radiation reaction force, and the term containing D describes the slowing of the particles due to the quasilinear diffusion, or, equivalently, due to the radiation reaction force of the anomalous Doppler resonance. To estimate the relative importance of these terms we consider a ratio

$$\frac{A\gamma^3}{Dmc} = \frac{\alpha\delta\gamma}{\beta} = \frac{4\delta\gamma}{3}\frac{R_B r_e}{r_L^2} \ll 1. \tag{36}$$

Neglecting the second term on the left-hand side of Eq. (36), we find

$$\frac{\partial f(p)}{\partial t} - \frac{2}{p^2}\frac{\partial}{\partial p}(pDm^2c^2E_k^2 f(p)) = 0. \tag{37}$$

If the cyclotron quasilinear diffusion has time to develop fully and reach a steady state, then the distribution function of the resonant particles is

$$f(p) \propto \frac{1}{pE_k^2}. \tag{38}$$

Next we turn to the equation describing the temporal evolution of the wave intensity (10). Neglecting the spontaneous emission term and the wave convection, we find

$$\frac{\partial E_k^2}{\partial t} = -\Gamma E_k^2 f(\gamma)_{\rm res}, \tag{39}$$

where

$$\Gamma = \frac{1}{f(\gamma)_{\rm res}} \sum_s \int d\mathbf{p}\; w(s,\mathbf{p},\mathbf{k}) \times \left[\hbar \left(\frac{\partial}{\partial p} + \frac{\cos\psi - (kv/\omega)\cos\theta}{p\sin\psi} \frac{\partial}{\partial \psi} \right) f(\mathbf{p}) \right], \tag{40}$$

and we introduced

$$f(\gamma)\gamma^2 d\gamma = f(p) p^2 dp. \tag{41}$$

We will estimate this growth rate for the emission along the external magnetic field for distributions (30) and (32). Neglecting $\partial/\partial p$ and assuming that $\psi^2 \ll 2\delta$ (so that most of the particles are moving with the superluminal velocity), we find, for $s=-1$,

$$\Gamma = \frac{\pi \omega_{p,\rm res}^2}{2\omega} \gamma_{\rm res}^2. \tag{42}$$

(It is important to note that in the limit $\psi^2 \ll 2\delta$ the growth rate does not depend on the scatter in pitch angles.)

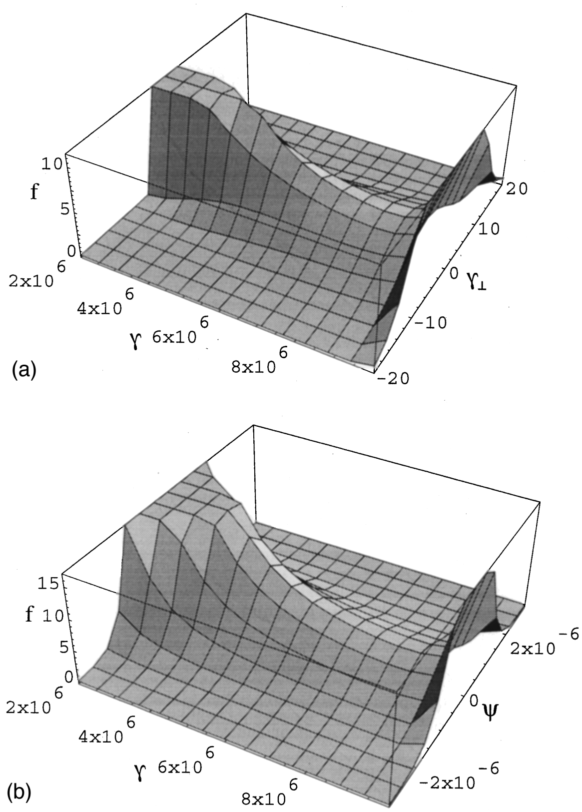

FIG. 1. Asymptotic distribution functions in $\gamma-\gamma_\perp$ (a) and $\gamma-\psi$ (b) spaces in arbitrary units for $\gamma_{\max}=10^7$. The spike at $\gamma=\gamma_{\max}$ is an artifact of the initial distribution function $f(\gamma)^0=\delta(\gamma-\gamma_{\max})$. The divergence at $\gamma=\gamma_{\max}$ is weak (logarithmic), and would be removed if the more realistic initial distribution function was used.

Equations (37) and (42) may be combined to a quasilinear expression

$$\frac{\partial}{\partial t}\left[f(\gamma) + \frac{2}{p^2}\frac{\partial}{\partial p}\left(\frac{pDm^2c^2E_k^2}{\Gamma} \right) \right] = 0, \tag{43}$$

which, after integration, gives

$$f(\gamma) - \frac{2}{\gamma^2}\frac{\partial}{\partial \gamma}\left(\frac{\gamma D E_k^2}{\Gamma} \right) = f^0(\gamma). \tag{44}$$

Neglecting the initial density of particles in the region of quasilinear relaxation, and using Eqs. (38) and (44), we can find a distribution function and the asymptotic spectral shape:

$$f(\gamma) = \frac{1}{2\gamma^3}\left(\frac{1}{\ln(\gamma_{\max}/\gamma)\ln(\gamma_{\max}/\gamma_{\min})} \right)^{1/2}, \tag{45}$$

$$E_k^2 = \frac{mc^2 \delta r_L \gamma^2}{2\pi r_e r_S^2}\left(\frac{\ln(\gamma_{\max}/\gamma)}{\ln(\gamma_{\max}/\gamma_{\min})} \right)^{1/2} = \frac{mc^4\delta^3}{2\pi\omega^2 r_e r_L r_S^2}\left(\frac{\ln(\gamma_{m\,\max}\omega r_L/(c\delta))}{\ln(\gamma_{\max}/\gamma_{\min})} \right)^{1/2}. \tag{46}$$

It is noteworthy that a simple power law distribution for the spectral intensity and distribution function cannot satisfy both Eqs. (38) and (44). The particle distributions function and the energy spectrum of the waves are displayed in Figs. 1 and 2.

We can now estimate the flux per unit frequency

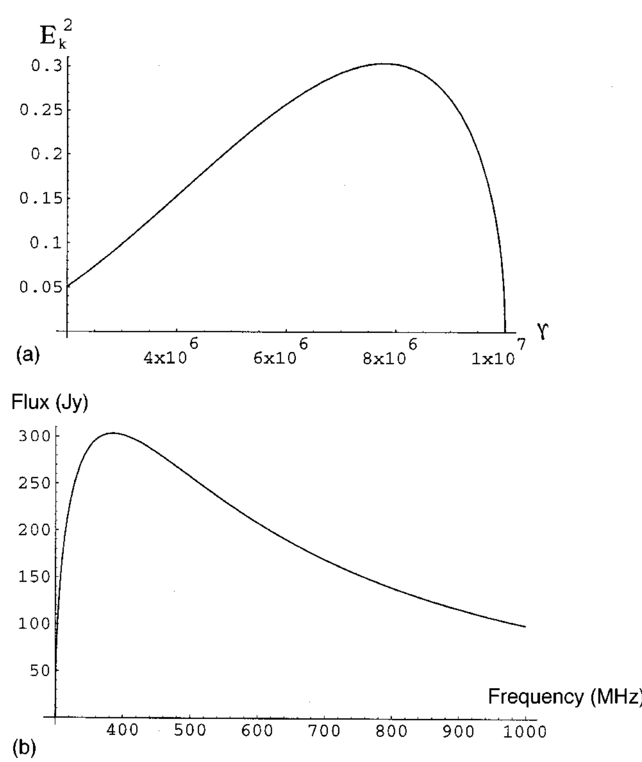

FIG. 2. Asymptotic one-dimensional energy density in the waves in γ space (a) in arbitrary units, and the predicted observed flux in Janskys (b).

$$F(\nu)=2\pi E_k^2=\frac{mc^4\delta^3}{\omega^2 r_e r_L r_S^2}\left(\frac{\ln(\gamma_{\max}\omega r_L/(c\delta))}{\ln\gamma_{\max}/\gamma_{\min}}\right)^{1/2}, \quad (47)$$

characteristic pitch angle

$$\psi_0=\delta\left(\frac{\pi R_B r_L}{r_S^2}\right)^{1/2}\left(\frac{\ln(\gamma_{\max}/\gamma)}{\ln(\gamma_{\max}/\gamma_{\min})}\right)^{1/4}\approx 10^{-6} \quad (48)$$

(which remarkably stays almost constant for a broad range of particles' energies, and also for different values of $\gamma_{\min}$), and the total energy density in the waves

$$E_{tot}=\int_{\nu_{\min}}^{\nu_{\max}}F(\nu)d\nu\approx\frac{mc^2\gamma_{\max}}{4\sqrt{\pi}\,r_e r_S^2\ln^{1/2}(\gamma_{\max}/\gamma_{\min})}, \quad (49)$$

This total energy can be compared with the kinetic energy density of the beam:

$$\frac{E_{\mathrm{tot}}}{\gamma_b mc^2 n_{GJ}}\approx\sqrt{\frac{\pi}{\ln(\gamma_{\max}/\gamma_{\min})}}. \quad (50)$$

This means that some considerable fraction of the beam energy can be transformed into waves.

We can also estimate the energy flux (47) at the Earth. Assuming that distance to the pulsar is $d\approx 1$ kpc, we find

$$F^{\mathrm{obs}}(\nu)\approx 300\ \mathrm{Jy}\left(\frac{\nu}{400\ \ \mathrm{MHz}}\right)^{-2}. \quad (51)$$

With time, the value of $\gamma_{\min}$ decreases as the particles are slowed down by the radiation reaction force. Since, at the given radius, particles with lower energies resonate with waves having larger frequencies, more energy will be transported to higher frequencies, hardening the spectrum. The lower frequency cutoff is determined by the initial energy of the beam. No energy is transported to frequencies lower than

$$\omega_{\min}=\frac{\omega_B}{\gamma_b\delta}. \quad (52)$$

This simple picture, of course, will be modified due to the propagation of the flow in the inhomogeneous magnetic field of the pulsar magnetosphere.

IV. CONCLUSION

In this work we investigated the saturation mechanism for the cyclotron Cherenkov instability of a beam in a inhomogeneous magnetic field. We showed that for typical parameters of the pulsar magnetosphere it is possible to reach a quasisteady state, in which the transverse motion of particles is determined by the balance of a radiation reaction force due to the emission at an anomalous Doppler effect and the force arising in the inhomogeneous magnetic field due to the conservation of adiabatic invariant. The resulting wave intensities are sufficient to explain the observed fluxes from radio pulsars.

ACKNOWLEDGMENTS

I would like to thank Roger Blandford, Peter Goldreich, and George Machabeli for their useful comments. I would also like to acknowledge the hospitality of the Abastumani Astrophysics Observatory during my visits. This research was supported by the NSF under Grant No. AST-9529170.

[1] J. Arons, Astrophys. J. **266**, 215 (1983).
[2] K. Kawamura and I. Suzuki, Astrophys. J. **217**, 832 (1977).
[3] J. G. Lominadze, G. Z. Machabeli, and A. B. Mikhailovsky, Fiz. Plazmy **5**, 1337 (1978) [Sov. J. Plasma Phys. **5**, 748 (1979)].
[4] A. Z. Kazbegi, G. Z. Machabeli, and G. I. Melikidze, Mon. Not. R. Astron. Soc. **253**, 377 (1991).
[5] M. Lyutikov, Ph.D. thesis, Caltech, 1998.
[6] J. G. Lominadze, G. Z. Machabeli, and V. V. Usov, Astrophys. Space Sci. **90**, 19 (1983).
[7] D. R. Lorimer *et al.*, Mon. Not. R. Astron. Soc. **273**, 411 (1995).
[8] M. Kramer *et al.*, Astron. Astrophys. **306**, 867 (1996).
[9] V. L. Ginzburg and V. Ya. Eidman, Zh. Eksp. Teor. Fiz. **36**, 1823 (1959) [Sov. Phys. JETP **36**, 1300 (1959).
[10] D. V. Sivukhin, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 7.